APPLICATIONS OF MARTINGALE SYSTEM THEOREMS

BY

LAURIE JAMES SNELL

B.S., University of Illinois, 1947
M.S., University of Illinois, 1948

AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN MATHEMATICS
IN THE GRADUATE COLLEGE OF THE
UNIVERSITY OF ILLINOIS, 1951

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Def. A stochastic process is a collection $[x_t, F_t, t \epsilon T]$ such that T is a subset of the real line extended by the adjunction of $+\infty$ and $-\infty$, F_t is a Borel field, and x_t is measurable with respect to the Borel field $\overline{F}_t$ consisting of sets of F_t or sets which differ from such sets by a set of probability one.

Def. Let x be a random variable such that $E[x]$ exists and let F be a Borel field. *Then the expected value of x relative to F* written $E[x \mid F]$ is any random variable y which is measurable with respect to $\overline{F}$, such that $E[y]$ exists and such that

$$\int_V y dpr = \int_V x dpr \qquad V \epsilon F$$

Def. A stochastic process $[x_t, F_t, t \epsilon T]$ is a *semimartingale* if

(a) $E[|x_t|] < \infty$ for all t

(b) $E[x_{t_2} \mid F_{t_1}] \geq x_{t_1}$

with probability one whenever $t_1 < t_2$.

a *martingale* if (b) is replaced by

(b′) $E[x_{t_2} \mid F_{t_1}] = x_{t_1}$

with probability one whenever $t_1 < t_2$.

A stochastic process $[x_t, F_t, t \epsilon T]$ is a *generalized semimartingale* or a *generalized martingale* if condition (a) in the above definition is weakened to require only that $E[x_t]$ exists for all t.

Doob (2) has proved the following theorem,

Theorem

Let $[x_n, 1 \leq n \leq m]$ *be a martingale and* $b(\omega)$ *be the number of upcrossings* by $x_1(\omega), \ldots, x_m(\omega)$ of $[r_1, r_2]$. *Then*

$$E[b] \leq \frac{E[|x_m|] + r_1}{r_2 - r_1}$$

In the above theorem $[x_n, 1 \leq n \leq m]$ is a stochastic process with F_n the field generated by $x_1, \ldots x_n$. The number of upcrossings $b(\omega)$ is defined as follows: Plot the points $x_1(\omega), \ldots . x_m(\omega)$ and join successive points by straight lines. Then $b(\omega)$ is the number of times the resulting graph goes from below r_1 to above r_2. In this paper the above theorem is generalized to obtain,

Theorem

Let $[x_n, F_n, 1 \leq n \leq m]$ *be a generalized semimartingale and* $b(\omega)$ *be the number of upcrossings by* $x_1(\omega), \ldots x_m(\omega)$ *of* $[r_1, r_2]$.

Then

$$E[b| F_1] \leq \frac{E[|x_m| | F_1] + r_1}{r_2 - r_1}$$

with probability one.

Using this theorem it is easy to prove the following convergence theorem due to Doob (1),

Theorem

Let $[x_n, F_n, n \geq 1]$ *be a semimartingale such that* $\sup E[|x_n|] < \infty$. *Then* $\lim x_n = x_\infty$ *exists with probability one and* $E[|x_\infty^n|] < \infty$.

The following theorem for convergence of generalized semimartingales is proved.

Theorem

Let $[x_n, F_n, n \geq 1]$ *be a generalized semimartingale. Then*

$\lim x_n = x_\infty$ *exists and is finite for almost all* ω *in the set* $[\inf_k \sup_{n \geq k} E[|x_n| \mid F_k] < \infty]$.

Def. If $X = [x_t, F_t, t \epsilon T]$ and $Y = [y_t, F_t, t \epsilon T]$ are stochastic processes having the same Borel fields, then $X < Y$ if for each t we have $Pr[x_t(\omega) \leq y_t(\omega)] = 1$.

Def. Let $X = [x_t, F_t, t \epsilon T]$ be a stochastic process. Then $Z = [z_t, F_t, t \epsilon T]$ is a *maximal generalized semimartingale less than X* if

(a) Z is a generalized semimartingale with $Z < X$

(b) If Z′ satisfies (a) then $Z' < Z$.

Def. Let $[x_n, F_n, n \geq 1]$ be a stochastic process. *An optional sampling random variable for X* is a random variable m such that the values of m are positive integers and $+\infty$, and

(a) $[m(\omega) = j] \epsilon F_j$

(b) $Pr[m(\omega) < \infty] = 1$.

In this paper the following problem is considered: we are given a stochastic process $X = [x_n, F_n, n \geq 1]$. With each optional sampling random variable m we associate the random variable $x(m, .)$ defined by $x(m,\omega) = x_m(\omega)$. Does there exist an optional sampling random variable, called a minimizing o.s. random variable, which mimimizes $E[x(m, .)]$? Arrow, Blackwell, and Girshick (1) have shown that under suitable hypothesis on the stochastic process X such a mimizing o.s. random variable does exist. In this paper the conditions of these authors are weakened and the existence of an o.s. random variable is shown to be a consequence of semimartingale theorems. In the case considered the mimizing o.s. random variable is defined as follows: the stochastic process X is such that there is a maximal generalized semi-

martingale $Z = [z_n, F_n, n \geq 1]$ with $Z < X$ and the mimimizing o.s. random variable m^* is defined by,

$$[m^*(\omega) = j] = [z_k(\omega) < x_k(\omega),\ k < j,\ z_j(\omega) = x_j(\omega)]$$
$$= \infty \text{ if not defined by above.}$$

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VITA

Laurie James Snell was born on January 15, 1925, in Wheaton, Illinois. After graduating from Wheaton High School in 1942, he entered the University of Illinois. He served in the Navy from 1944 to 1946. Upon discharge from the Navy, he returned to the University of Illinois and received a Bachelor of Arts degree in 1947 and a Master of Science degree in 1948. He held a University fellowship during the academic years 1947-48, 1948-49, and 1949-50. He was an assistant in Mathematics during the academic year 1950-51.

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